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A Map Technique for Identifying Variables of Symmetry

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This paper presents a new map technique for identifying symmetrizable functions. The technique greatly reduces the work in ascertaining symmetricity, and it is unique in being also applicable to completely or incompletely specified functions which:

(*i*) *Contain imbedded symmetrizable function(s).*
(*ii*) *Are the complement of a function of type* (*i*).
(*iii*) *Contain an imbedded function of type* (*ii*).

Discussion of the technique and its extensions is included.

I. INTRODUCTION

Recognition of symmetry in circuit design often can drastically reduce the problem of finding the least expensive circuit configuration. Multi-output circuits frequently have a symmetric circuit as a common portion so no single error will result in a wrong output.

As a consequence, numerous papers and chapters of books have presented recognition of symmetry in a switching function.[1–34] However, none of these articles has presented a technique that is simple to apply and has natural extensions to accommodate both completely and incompletely specified functions that are almost symmetrizable.* This paper presents such a technique.

Caldwell[1,2] has demonstrated a technique using Karnaugh maps for recognizing symmetrizable functions (SF's) of three or four variables, and has also demonstrated a procedure for extending this to functions of more variables. The extension requires the use of a large number of maps and the use of an expansion theorem a multiplicity of times. The Caldwell technique requires mapping all possible submaps in four of the variables.

* See Section II for definitions.

Few autbors have dealt with any subset of almost SF's (ASF's), and even fewer have tried to give practical examples of a technique for their solution.

Born and Scidmore[3] have dealt with the special subset of ASF's commonly called partial symmetric functions. They have solved a function in six variables for whicb, by their definition, a minimum solution exists and a resulting realization is sbown in Fig. 1.* This same example has been solved by tbe technique presented in this paper and tbe resulting circuit, shown in Fig. 2, is significantly more economical.†

In general, other techniques attempt to ascertain symmetry and find tbe center of symmetry (COS) simultaneously. The technique presented here first uses a set of overlapping maps to ascertain wbat the COS must be if the function is an SF (or ASF) and then verifies whether tbe function is symmetric (or almost symmetric) about tbat COS.

II. METHOD

2.1 Theory

Shannon first stated the definition of a symmetric function as follows: "A function of n variables $L_1, L_2, L_3, \cdots, L_n$ is said to be symmetric in these variables if any interchange of the variables leaves the function the same. . . . Since any permutation of variables may be obtained by successive interchange of two variables, a necessary and sufficient condition that a function be a symmetric is tbat any interchange of two variables leaves the function unaltered."[4]

The nomenclature for a symmetric function bas been established by prior usage.[5]

A function f is called an SF if and only if f is equivalent to some function g where g is a symmetric function. Two functions are considered equivalent when one may be obtained from the otber by complementing some variables.

When a function f is an SF, the variables of g (any symmetric function equivalent to f) and their complements are called COS's. Sucb a pair defines an axis of symmetry uniquely specified by eitber member of tbe pair. Althougb any SF bas at least two COS's, the function is not symmetric in the same degree (tbe subscript in standard symmetric notation) about tbe two centers. In fact, if a function can be represented as m out of n about one COS, then it is $(n - m)$ out of n about tbe complemented COS.

* This circuit realization could have been as easily done with FET's in MOS technology. Each contact would he replaced with a single FET.

† The details of this are presented in the appendix.

RELAY	CONTACT LOAD
A	4T
B	5T
C	5T,1B
D	5T,1B
E	5T
F	4T
TOTAL	28T,2B

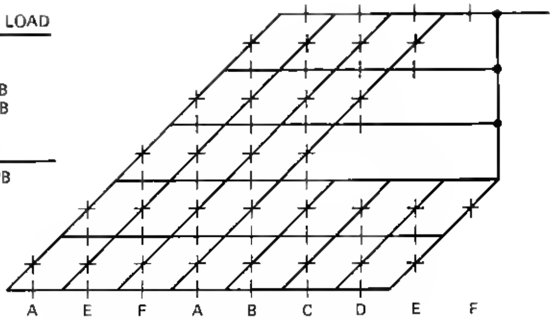


Fig. 1—Simplified circuit for Born and Scidmore problem by Born and Scidmore technique.

A function is an ASF if it

(*i*) Contains one or more imhedded SF's, i.e., the function can be expressed as the sum (ORing) of two or more functions where one or more functions are SF's.

(*ii*) Is the complement of a function containing one or more imbedded SF's.

(*iii*) Can be expressed as the sum of two functions, one of which is the complement of a function containing one or more imbedded SF's.

Any function is an ASF if the limits are stretched far enough, hut to henefit from symmetricity the number of imbedded SF's should he small and the terms not included in the SF's should be relatively few.

RELAY	LOAD
A	3T
B	2T
C	1T
D	2T,1B
E	2T
F	1T
TOTAL	11T,1B

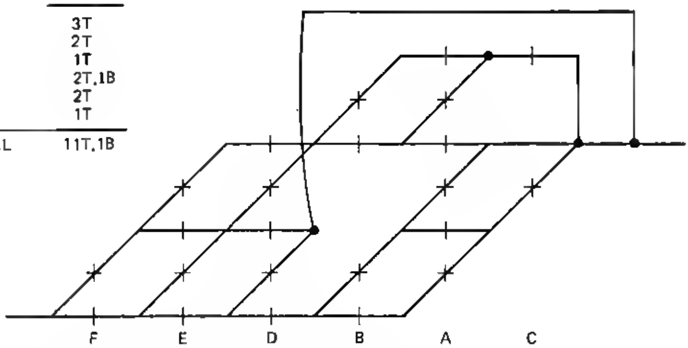


Fig. 2—Simplified circuit for Born and Scidmore problem by author's technique.

Any function fits into one of three categories:

(*i*) The function is not symmetrizable.
(*ii*) The function is symmetrizable about all points.
(*iii*) The function is symmetrizable about exactly two centers of symmetry.[6]

The first category is not the subject of this paper.

The second category consists of only four members, the "zero" function, the "one" function, and the odd and the even parity functions. The first two functions are trivial, while the parity functions are very specific functions that have been the subject of numerous papers.* These functions have exactly 2^{2-1} minterms for a function of n variables, and no minterm differs from any other minterm in the state of an odd number of variables.

The third category is of practical interest, being the class of SF's that are not just trivially symmetric. It is also the large majority of SF's with more than three variables ($n > 3$), since this category has $2^{n+1} - 4$ members.

A necessary and sufficient condition for a function to be symmetric about a specific COS may be expressed as follows: If one minterm matches the COS in exactly m out of the total of n variables, then exactly ${}_mC_n$† minterms must match that COS in exactly m out of n variables. This is a direct result of the definition of an SF.

Thus, a simple test exists for ascertaining whether a given function is an SF about a specific COS.‡ The problem then is determining what a COS must be if the function is an SF. Consequently, rather than exhaustively testing a function for symmetry, our technique first finds what the COS must be if the function is an SF and then verifies the actual symmetry about that point.

A direct result of the theorem[11]

$$S_a^n(L_1, L_2, L_3, \cdots, L_n) = \sum_{j=0}^{m} S_j^m(L_1, L_2, L_3, \cdots, L_m) \cdot S_{a-j}^{n-m}(L_{m+1}, L_{m+2}, \cdots, L_n)\text{§}$$

is that a function f is an SF in $L_1, L_2, L_3, \cdots, L_n$ only if a specific subset of the minterms is an SF in $L_1, L_2, L_3, \cdots, L_m$ for $m \leqq n$. Thus,

* Garner (Ref. 7) is one of many who have studied this function.

† ${}_mC_n$ is the number of combinations of n things taken m at a time and equals $n!/[(n-m)!\,n!]$.

‡ In essence, this is the same test that McCluskey (Refs. 8 and 9), Marcus (Refs. 5 and 10), and others have used.

§ $S_a^n(L_1, L_2, L_3, \cdots, L_n)$ is standard symmetric notation for "symmetric a-out-of-n function of variables $L_1, L_2, \cdots, L_n$."

for any SF, a COS in n variables when all other variables are held constant is a subset of the total COS of the function.

We have shown that a necessary condition for an SF is that the function be symmetrizable in subsets of the variables. This, however, is not a sufficient condition even if the subsets encompass all variables, as we can show by an example. The eight-variable function $f = L_1L_2L_3L_4S_3^4(L_5, L_6, L_7, L_8)$ is symmetrizable in both the four most and the four least significant variables, but is not an SF in all eight variables. Even the fact that the function is symmetrizable in overlapping subsets of variables is not a sufficient condition, as we can demonstrate by the seven-variable function $f = S_6^7(L_1, L_2, \cdots, L_7) + (L_1L_2L_3L_4'L_5'L_6'L_7')$. Thus, the possibility exists that a function is not an SF, even though a composite of the COS's of subsets can be found. However, if COS's are found for subsets of variables where the group of subsets includes all variables in the function, then a COS of the function (if it exists) must be the composite of the COS's of the subsets.

Since each COS has a mate (the point where all variables are the complements of the variables of the first COS), a COS of the function must be a composite involving one or the other COS. The ambiguity as to which COS of each pair to select can be resolved by having each set of variables overlap another set in at least one variable. Thus, if there are n variables and each subset selected has k variables, then at least $\lceil \overline{(n-1)}/\overline{(k-1)} \rceil$ subsets are required to find the COS.*

S. H. Caldwell[1,2] has demonstrated a technique using Karnaugh maps for recognizing symmetry in functions of three or four variables. We review it here.

Any single square on a Karnaugh map represents an SF of n out of n, where n equals the number of variables. Thus, on a three-variable map, each square is an SF of three out of three [written $S_3^3(L_1, L_2, L_3)$] of some set of variables L_1, L_2, and L_3. As an example, the square α in Fig. 3 is $A'BC'$ and is the *SF* $S_3^3(A'BC')$.

On a Karnaugh map, only one variable changes state in going from one square to an adjacent square. Thus, the four adjacent squares match the center square in two out of three variables.† Similarly, any squares that are two squares from the given square match it in one variable, etc. Thus, the squares labeled 2, 1, and 0 are S_2^3, S_1^3, and S_0^3 of

* $\lceil\overline{\quad}\ \overline{\quad}\rceil$ Notation for the least integer equal to or greater than the argument.
† To see the pattern, it is necessary to extend the map-repeating columns and rows, as shown in Fig. 3.

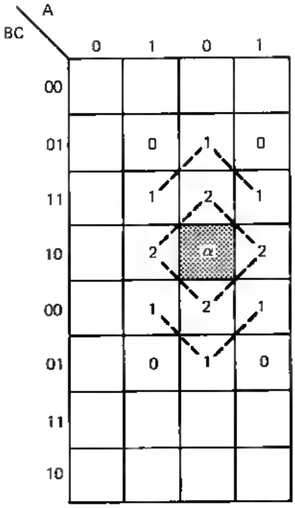


Fig. 3—Symmetry in a three-variable Karnaugh map.

square α, respectively. Note that the coordinates of square α are the variables of symmetry L_1, L_2, L_3, etc. (i.e., the COS).

Expanding this to four variables represents the situation shown in Fig. 4, where the 3's represent S_3^4 of square β, the 2's represent S_2^4 of square β, etc.

Since each SF of three or four variables yields a distinctive pattern, pattern recognition permits recognizing any SF of three or four vari-

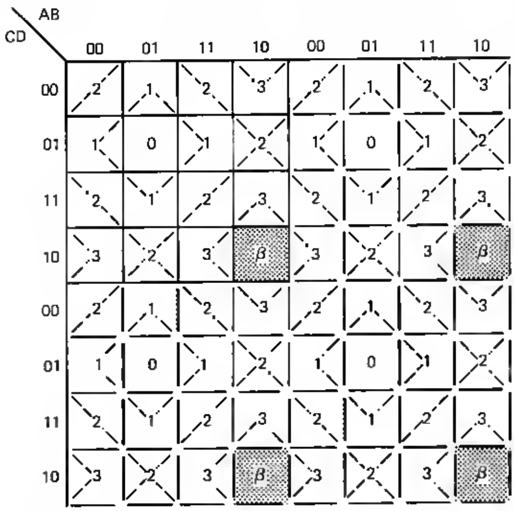
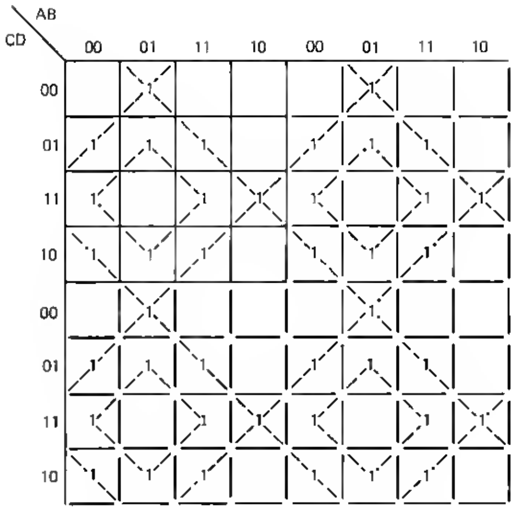


Fig. 4—Symmetry in a four-variable Karnaugh map.



$S^4_{2,3}$ (A'BCD) = S^4_2 (A'BCD) + S^4_3 (A'BCD)

Fig. 5—Sum of two fundamental symmetrics.

ables, and the COS identifies the variables of symmetry. Note that $S^4_0(L_1, L_2, L_3, L_4) = S^4_4(L'_1, L'_2, L'_3, L'_4)$, etc.

The identification of an SF whicb is tbe sum of more than onc fundamental SF[12] in thc samc variablcs is depicted in Fig. 5. This method can be cxtendcd to morc than four variables, as explaincd bclow.

At this point, wc depart from Caldwell's tcchnique. Note that Caldwcll has presentcd a means of expanding his method to morc than four variables (in theory, to any number). However, for n variablcs, his tcchnique requircs the usc of 2^{n-4} Karnaugh maps and $2^{n-4}-1$ applications of an expansion theorem.[13]

Since a technique exists for handling subsets of four or less variablcs, four can be substituted for k in tbe expression $\lceil\overline{(n-1)/(k-1)}\rceil$, yielding $\lceil\overline{(n-1)/3}\rceil$ as the minimum number of subsets required to find the COS of a function of n variables, if it exists. By appropriate selection of subsets, it is possible to require only the use of the minimum number of subsets.

2.2 Technique

First, wc list thc terms in decimal notation* where unprimed and primed variables are rcprcsented by binary ones and zeros, respectively.

* The use of decimal notation is not essential, but it speeds up the mapping and selection of terms which differ only in a specific set of successive variables.

Then, some four-variable Karnaugh maps are drawn. In general, one such map covering the lowest decimal numbers in the function forms a good starting point. Figure 6 presents a review of the decimal notation for Karnaugh maps with six variables.

Once the Karnaugh map has been plotted, we can try to identify an SF on it. If the submap is quite full of ones, checking the remaining zeros for symmetry may be easier. The SF of $S^4_{0,1,2,4}(A, B, C, D')$ shown in Fig. 7 is easier to identify as the complement of $S^4_3(A, B, C, D')$.

If no COS is found on a map, the function is not an SF. If a COS is found, then at least one of the four variables involved plus not more than three new variables is plotted in a similar fashion.* The resulting terms are plotted on a new four-variable map, and the COS (if it exists) is found.

This technique is repeated as many times as is necessary to account for all variables. Since the minimum number of maps is $\lceil (n-1)/3 \rceil$, the minimum (and usual) number of maps for nine variables is three. Figure 8 tabulates the minimum number of extended four-variable Karnaugh maps required by this technique as compared with the Caldwell-Grea technique. Once the various COS's are found, they are combined to form the COS of the whole function, complementing all the variables in any COS required to make the overlap variables match. This possibility exists since either of two COS's could be selected—i.e., $S^4_3(ABCD) = S^4_1(A'B'C'D')$.

Once the potential COS is found, each term of the function is compared with it to see if a complete SF is represented.

The only restriction on the selection of a subset of minterms for which all variables are fixed except a specific four is that the subset must have at least one member and may not have either 8 or all 16 members.†

The reason for the eight-term restriction is that eight terms in four variables represent either the odd or even parity function, which are

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* A suggested technique for this is to divide the decimal value of each term by 2 to include one new variable, by 4 to include two new variables, and by 8 to include three new variables. If insufficient terms are found by this technique, any number less than the divisor can be subtracted from each term's value prior to the division. Division by 8 selects those terms ending in 000 and discards the last three terms. Subtraction of 2 followed by division by 8 selects the leftmost $n - 3$ bits of those terms ending in 010, etc.

† $S^{10}_{0,1,2,3,4}$ $(ABCDEFGHIJ)$ has one full subset for any four variables (i.e., the subset where all fixed variables are zero although all other subsets in those variables are not full). There are no terms greater than 640 in the above function, since all such terms require at least five ones.

AB = 00

EF \ CD	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

AB = 01

EF \ CD	00	01	11	10
00	16	20	28	24
01	17	21	29	25
11	19	23	31	27
10	18	22	30	26

AB = 10

EF \ CD	00	01	11	10
00	32	36	44	40
01	33	37	45	41
11	35	39	47	43
10	34	38	46	42

AB = 11

EF \ CD	00	01	11	10
00	48	52	60	56
01	49	53	61	57
11	51	55	63	59
10	50	54	62	58

Fig. 6—Decimal notation for Karnaugh maps.

CD \ AB	00	01	11	10	00	01	11	10
00	1	1	0	1	1	1	0	1
01	1	1	1	1	1	1	1	1
11	1	1	0	1	1	1	0	1
10	1	0	1	0	1	0	1	0
00	1	1	0	1	1	1	0	1
01	1	1	1	1	1	1	1	1
11	1	1	0	1	1	1	0	1
10	1	0	1	0	1	0	1	0

THE 0's REPRESENT S_3^4 (ABCD')

THEREFORE THE 1's REPRESENT $S^4_{0,1,2,4}$ (ABCD')

Fig. 7—Identifying a large symmetric.

NUMBER OF VARIABLES	NUMBER OF EXTENDED FOUR VARIABLE KARNAUGH MAPS REQUIRED		NUMBER OF APPLICATIONS OF EXPANSION THEOREM	
	AUTHOR	CALDWELL–GREA	AUTHOR	CALDWELL–GREA
4	1	1	0	0
5	2	2	0	1
6	2	4	0	3
7	2	8	0	7
8	3	16	0	15
9	3	32	0	31
10	3	64	0	63
11	4	128	0	127
12	4	256	0	255
13	4	512	0	511
14	5	1024	0	1023
15	5	2048	0	2047
16	5	4096	0	4095
17	6	8192	0	8191
18	6	16,384	0	16,383
19	6	32,768	0	31,768
20	7	65,536	0	65,535

Fig. 8—Comparison of author's technique and Caldwell-Grea technique.

symmetrical about all possible minterms in four variables. In general, the selection of another set of values for the fixed variables results in a map without eight terms.*

2.3 *Illustrative example*

We test here a 10-variable function for symmetry. Trying all 2^{10} possible combinations of terms or plotting the 64 four-variable Karnaugh maps and mathematically combining them are unattractive. On the other hand, a 10-variable function is not unwieldy by this technique.

Let us consider the function in variables A through J where $F = \sum$ (13, 21, 25, 28, 31, 37, 41, 44, 47, 49, 52, 55, 56, 59, 62, 93, 109, 117, 121, 124, 127, 157, 173, 181, 185, 188, 191, 253, 285, 301, 309, 313,

* $S^{10}_{1,3}$ $(ABCDEFGHIJ)$ will exhibit apparent parity for only those subsets where all fixed variables have a value of zero.

316, 319, 381, 445, 541, 550, 557, 565, 569, 572, 575, 637, 701, 706, 829, 834, 898, 960, 963, 966, 970, 978, 994).

Figure 9 is a worksheet listing the terms of the function and the terms to be plotted on Karnaugh submaps. The appropriate Karnaugh submaps are shown in Fig. 10.

For the subset in which A through F are zero, the only term is 13 and thus it must be a COS of the four variables $GHIJ$ (1101). Next, we find terms which fit the pattern 000XXXX000. Division by 8 yields one such term, i.e., 7, and so the COS in the four variables $DEFG$ must be 0111.

Next, we find a submap of the variables $ABCD$ by selecting those terms ending in 111000 (i.e., subtracting 7 from the terms found by the first division and then dividing by 8). The only resulting term is 0000. Thus, the COS of the whole function (if it exists) must be a composite of $ABCD$ = 0000 and $DEFG$ = 0111, and $GHIJ$ = 1101, i.e., 0000111101.

All that remains is to verify whether or not the function is an SF about this COS. Consequently, the COS is filled in at the head of the "MATCH" column on the worksheet and each individual term is

n	A n/8	(A-7)/8	TERM	MATCH	n	A n/8	(A-7)/8	TERM	MATCH
13					301				
21					309				
25					313				
28					316				
31					319				
37					381				
41					445				
44					450				
47					541				
49					557				
52					565				
55					569				
56	7	0			572				
59					575				
62					637				
93					701				
109					706				
117					829				
121					834				
124					898				
127					960	120			
157					963				
173					966				
181					970				
185					978				
188					994				
191									
253									
285									

Fig. 9—Worksheet for sample problem.

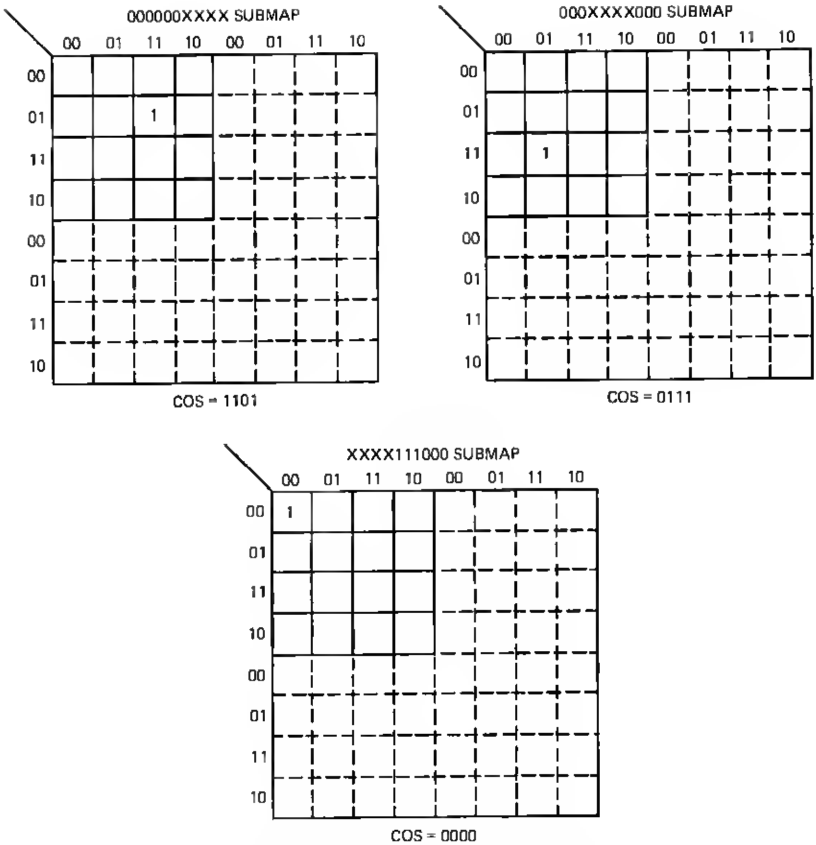

Fig. 10—Karnaugh submaps for sample problem.

compared with this pattern and the number of matching variables recorded. Figure 11 presents the completed worksheet. Next, the number of matches is compared with the number required for an SF, i.e., ${}_mC_n$. For $m = 8$ and $n = 10$, ${}_mC_n = 10!/(8! \times 2!) = 45$ and for $m = 1$ and $n = 10$, ${}_mC_n = 10!/(1!\,9!) = 10$.

Since 45 terms match on eight variables and 10 terms match on one variable, the function is an SF, i.e., $S^{10}_{1,8}(61)$.*

A technique has been presented that is simple to use manually and requires no extensive memorization of a routine such as required by the Marcus-McCluskey method.[10,8] However, the real power of the

* This is the decimal notation abbreviation for $S^{10}_{1,8}$ $(A'B'C'D'EFGHI'J)$.

n	A n/8	(A-7)/8	TERM	MATCH 0000111101	n	A n/8	(A-7)/8	TERM	MATCH 0000111101
13			0000001101	8	301			0100101101	8
21			0000010101	8	309			0100110101	8
25			0000011001	8	313			0100111001	8
28			0000011100	8	316			0100111100	8
31			0000011111	8	319			0100111111	8
37			0000100101	8	381			0101111101	8
41			0000101001	8	445			0110111101	8
44			0000101100	8	450			0111000010	1
47			0000101111	8	541			1000011101	8
49			0000110001	8	557			1000101101	8
52			0000110100	8	565			1000110101	8
55			0000110111	8	569			1000111001	8
56	7	0	0000111000	8	572			1000111100	8
59			0000111011	8	575			1000111111	8
62			0000111110	8	637			1001111101	8
93			0001011101	8	701			1010111101	8
109			0001101101	8	706			1011000010	1
117			0001110101	8	829			1100111101	8
121			0001111001	8	834			1101000010	1
124			0001111100	8	898			1110000010	1
127			0001111111	8	960	120		1111000000	1
157			0010011101	8	963			1111000011	1
173			0010101101	8	966			1111000110	1
181			0010110101	8	970			1111001010	1
185			0010111001	8	978			1111010010	1
188			0010111100	8	994			1111100010	1
191			0010111111	8					
253			0011111101	8					
285			0100011101	8					

Fig. 11—Completed worksheet for sample problem.

technique is that it can readily be extended to cases that are not pure SF's, while the other methods cannot be so extended. The reason the new technique can be extended is hecause it depends on pattern recognition at which humans are adept. Thus, patterns can be discerned in spite of extraneous data, i.e., "noise," while a technique which rigorously uses all data in a prescrihed functional relationship cannot sort out the desired data. Obviously, an SF can he so ohscured by "noise" as to be unrecognizahle. However, the cases of the most value are those which have only limited ohscuring terms.

In the next section, some extensions of this technique are presented.

III. EXTENSIONS OF THE TECHNIQUE

Since the technique may he extended to cover a variety of situations, we discuss some specific extensions here. The author has worked problems for each of the discussed extensions, and at least one of each has been presented in unpublished memoranda, while one example of a composite function is solved in the appendix.

For the purpose of this paper, a function is a multiple SF if it is the sum of two or more SF's whose COS's are not the same or complements of each other. Thus, the function $f = S_2^5(ABCDE') + S_3^5(A'BCDE)$ is said to be a multiple SF. Since any minterm of a function of n variables is the SF S_n^n (minterm), any function of m minterms can be represented as the sum of not more than m SF's. However, the useful cases are functions that are the sum of considerably less SF's than minterms, such as the above example.

An incompletely specified function is a function that contains at least one minterm for which transmission is neither required nor forbidden (don't-care terms).

A function is called an incomplete SF if all but a few minterms required for an SF are present.

An overly complete SF is a function composed of an SF plus additional minterms. Any function can be forced to fit this mold, but the cases of interest are those in which almost all the minterms are included in the SF.

Approximate SF's are functions that are not truly SF's but that differ only slightly from an SF. Incomplete SF's are included in approximate SF's, as are overly complete SF's. However, approximate SF's also cover functions that are the sum of incomplete SF's and some additional minterms.

Obviously, any function fits into the category of an approximate SF, but the useful cases are those whose deviation from an SF is slight. For example, the function $f = S_3^8(ABC'DE'FG'H) + AB'C'D'EF'GH' - ABC'D'EF'GH'$ would fit this category. Functions like this can often be simply constructed by modifying the SF.

A partial symmetric has been defined as a function which is an SF in some but not necessarily all variables.[14] Thus, SF's are a subset of partial symmetrics. However, usually a partial symmetric refers to one which is not an SF in all variables.

Both multiple SF's and overly complete SF's are examples of functions containing imbedded SF(s). An incomplete SF is a function whose complement contains an imbedded SF, while approximate SF's are the sum of two functions, one of which is the complement of a function containing an imbedded SF.

3.1 Overly complete SF's

In many cases, a function can be expressed as an SF plus certain additional terms. That is to say, the function has an imbedded SF. The

cases of interest are those in which the number of additional terms is small, since a symmetric design can be used for most terms and then the additional terms can be treated individually. In some cases, part of the additional terms can be combined directly with the SF.

The technique is basically the same as it is for a pure SF. However, more maps may be required because of the masking effect of potential extraneous minterms. An extra minterm on a map could result in an ambiguity as to where the COS is. Worse than this, a single extraneous term could be the only term which appears on a specific submap. This could lead to the selection of a false COS for the entire function and an indication that the function did not contain an SF.

Because of this, it is best to select two submaps in the same four variables and get a concurrence as to the COS of the four variables. It is, of course, possible to find in some functions pairs of submaps in the same variables which do not concur on a single COS. Then a third submap is required in those same four variables. In the author's experience, this seldom occurs.

Theoretically, it is possible to have to continue making more submaps without actually determining what the COS of the four variables must be. However, the cases of the most value are those cases that require the least maps. In practice, if concurrence on a COS cannot be found by the use of four submaps in the same four variables, no very useful SF exists within the function. Also, if any one submap has more than three terms, it is usually possible to determine the COS of the four variables.

When dealing with an SF, we stated that a submap with eight terms represents one of the two parity functions. The parity function is recognized by the fact that, on a Karnaugh map, no two adjacent squares have the same value (zero or one). If eight terms appear on the submap and this relationship is not true, then the COS can usually be readily determined.

As before, once the various COS's are found, they are combined to form the point which must be the COS if the function is primarily composed of an SF. Then each term in the original function is compared with it to determine if all the terms of an SF are represented and which, if any, additional terms are included in the function.

3.2 Multiple SF's

It is usually possible, if there are two or more SF's imbedded in a function, to find each of them. The technique is based on the principle

used in finding an SF. Again, submaps are selected for the sets of four variables, using more than one submap in the same four variables if necessary. In practice, there is often less confusion when the additional terms do form an SF. Usually it is possible to see more than one COS in a single submap unless the two SF's both have the same COS in the chosen variables. However, since it is possible to have two SF's which have the same COS in four variables although not in all the function variables, the appearance of only one COS in a submap does not rule out the possibility of having more than one SF. Also, since in some submaps no terms of one SF may appear, if a function is found to have some isolated terms, these terms should be investigated to see if they (plus perhaps some terms in the discovered SF) form another SF.

At any point in the analysis of the function that more than one SF appears to be found on the same submap, the remainder of the analysis should be done on the basis of a supposed multiple SF. When two (or more) COS's are found that are not identical or mates, each COS is used as a basis for determining the minterms to be used in a submap in the next selected set of variables. In general, selection of two new variables and two old variables works better when a multiple SF is suspected, since the new set of variables must contain enough information to make possible the selection of the appropriate COS. As long as the variables to be held fixed while new submaps are defined differ in at least one position, different submaps can be defined for finding the next COS, i.e., the submaps 00XXXX11 and 00XXXX01 are different and usually will give COS's that can be readily correlated with the COS's found in the lowest-order submaps (perhaps 1011 and 0001).

3.3 Incomplete SF's

An incomplete SF is a function that lacks a few specific terms of being an SF. Thus, it is the complement of a function with an imbedded SF. Such a function can be expressed as one of the three following functions.

(*i*) A modified symmetric circuit.
(*ii*) A symmetric circuit ANDed with another circuit which blocks those terms supplied in the symmetric circuit that are not part of the total function.
(*iii*) A combination of the above.

In the first case, such a modified symmetric circuit would be very efficient. In the second and third cases, the value of finding the SF decreases as the complexity of the blocking circuit increases.

The technique for finding an incomplete SF is similar to that used in previous examples. The major point of difference is that some terms needed to complete the SF are not in the original function, and hence they must be added to the function to form an SF and then blocked by circuit changes.

Finding the COS may require addition of certain terms to the function. Once the proposed COS is found, additional terms may be required to complete the SF.

All terms added to find the COS or to complete the SF must be recorded so the effect of these terms can be deleted in the realization of the function.

3.4 Incompletely specified functions

An incompletely specified function is one that has at least one minterm for which the function is undefined. The use of some terms for which the function is undefined in conjunction with those terms for which the function is defined often permits simpler circuit configurations than could be achieved if all these extra terms are ignored (tacitly forced to a definition of no transmission states). The literature abounds with references to use of these terms in conjunction with Karnaugh maps. Although the use of these terms to aid in completing an SF has been previously ignored by most authors, some work has been done by Arnold and Lawler.[6]

Since the method described here depends on pattern recognition, the use of some terms to complete map patterns is straightforward. Once some (or none) of the undefined minterms have been used to ascertain what the COS of the function must be, then all terms for which transmission is required plus all terms for which the function is undefined are compared with the COS. Those undefined terms which have the same order of symmetry as the required terms are then used to test for the symmetricity of the function and, if sufficient terms are found, these terms are used to transform the function into an SF.

3.5 Approximate SF's

An approximate SF is a function that differs only slightly from a true SF. The cases of greatest interest are those that have a few terms not in the SF and are missing some terms needed to complete the SF. Thus, we can think of this as a combination of an overly complete SF and an incomplete SF. The most interesting problem solutions in these functions occur where the extra and the missing terms can be paired to result in only a minor modification to the true SF.

Here, as before, the basic attack is to find a COS about which the majority of the terms must be symmetric if the function is to approximate an SF. Since both missing and extra terms may occur, it may be necessary to ignore some terms in attempting to find a COS and it may also be necessary to temporarily add to the function certain other terms to complete the symmetry in some subset of variables. Once a proposed COS is found, certain terms may not be symmetric about that COS, and there may not be enough terms symmetric about the proposed COS to complete the SF.

3.6 *Composite function*

A composite function is a function that may be an incompletely specified function and that may contain one or more imbedded, incomplete, or complete SF's. In other words, the function can be almost any function.

The procedure is basically the same as has already been used. However, because of the possible complexity of the function, some feature may be obscured and, even though a COS for part of the terms is found, it may be desirable to repeat the process, treating all terms in the first-found incomplete or complete SF as don't-care terms for subsequent analysis. This procedure can be repeated until the work entailed is not warranted by the number of remaining terms not identified with an SF. The Born-Scidmore[3] problem considered in the appendix is an example of a composite function.

IV. CONCLUSIONS

A technique has been presented here for isolating SF's (complete or incomplete) in the presence of other SF's or other terms, or in incompletely specified functions.

Since the method depends basically on pattern recognition, at which humans excel, as opposed to value manipulation, at which they do not excel, this technique is especially useful for manual use. However, since the number of patterns is quite restricted, such a technique could be implemented by a computer program.

Since symmetrics are powerful tools in implementing functions, the recognition of SF's within functions can greatly reduce the work in the synthesis of functions.

V. ACKNOWLEDGMENTS

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APPENDIX

In Reference 3, Born and Scidmore have transformed a partial symmetric in six variables into an SF (pure symmetric) in nine variables. Specifically, the function they examined was $F = \sum$ (1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 24, 25, 26, 28, 33, 34, 35, 36, 37, 38, 40, 41, 42, 44, 48, 49, 50, 52, 56, 57, 58, 60). The minimum SF they presented was $S^9_{2,3,4,5}$ $(A, B, C, D, D, E, E, F, F)$. This SF can be represented as shown in Fig. 12 and, as it is shown, requires 34 transfers plus 2 break contacts with one relay requiring 10 transfers and 2 breaks.

However, this circuit can be simplified by standard techniques. The resulting simplified circuit is shown in Fig. 13, which uses 28 transfers and 2 breaks with the largest single-contact load being 5 transfers and 1 break.

This problem can also be solved by the author's technique. Figure 14 is the worksheet and Fig. 15 the first set of Karnaugh submaps. The 00XXXX submap shows two axes of symmetry, one with one center at 0111 and the other with one center at 1111. Subtracting 3 and dividing by 4 yields four terms for the XXXX11 submap. These four terms almost identify a center of symmetry at 0000 (or 1111). Either one term is missing (0001) which would have come from a minterm 7 or the term 0000 is actually that term shifted out of place by a modification of the symmetric. Either case argues for the use of this 1111 as a center of symmetry. When this is combined with the centers of symmetry in 00XXXX, one match is found, namely, 111111. Thus, the terms are compared to this center of symmetry and 15 terms are found to match it in two variables. Since this is the value of ${}_2C_6$, the function contains $S^6_2(63)$. Next, the remaining terms are plotted on the

RELAY	CONTACT LOAD
A	1T
B	2T
C	3T
D	9T
E	10T,2B
F	9T
TOTAL	34T,2B

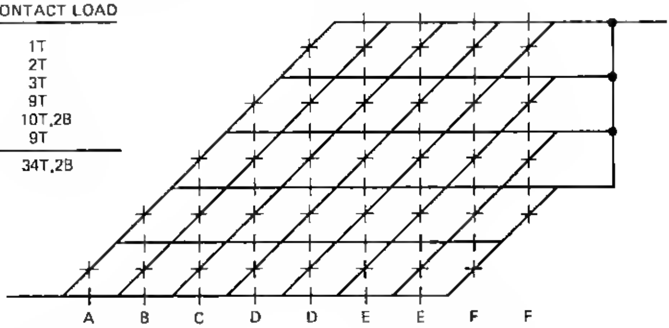

Fig. 12—Symmetric for $S^9_{2,3,4,5}$ $(A, B, C, D, D, E, E, F, F)$.

RELAY	CONTACT LOAD
A	4T
B	5T
C	5T,18
D	5T,18
E	5T
F	4T
TOTAL	28T,28

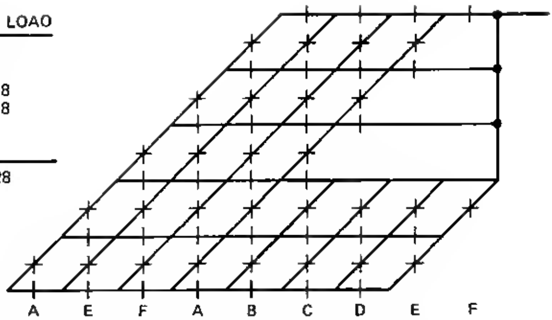

Fig. 13—Simplified symmetric for $S^9_{2,3,4,6}$ $(A, B, C, D, D, E, E, F, F)$.

N	(N-3)/4	TERM	MATCH 111111	MATCH 110111	TERMS MATCHING NEITHER IN TWO VARIABLES
1		000001	1	2	
2		000010	1	2	
3	0	000011	2	3	
4		000100	1	2	
5		000101	2	3	
6		000110	2	3	
9		001001	2	1	
10		001010	2	1	
11	2	001011	3	2	
12		001100	2	1	
13		001101	3	2	
14		001110	3	2	
17		010001	2	3	
18		010010	2	3	
19	4	010011	3	4	✓
20		010100	2	3	
21		010101	3	4	✓
22		010110	3	4	✓
24		011000	2	1	
25		011001	3	2	
26		011010	3	2	
28		011100	3	2	
33		100001	2	3	
34		100010	2	3	
35	8	100011	3	4	✓
36		100100	2	3	
37		100101	3	4	✓
38		100110	3	4	✓
40		101000	2	1	
41		101001	3	2	
42		101010	3	2	
44		101100	3	2	
48		110000	2	3	
49		110001	3	4	✓
50		110010	3	4	✓
52		110100	3	4	✓
56		111000	3	2	
57		111001	4	3	✓
58		111010	4	3	✓
60		111100	4	3	✓
	MISSING TERMS				
16		010000	1	2	
32		100000	1	2	

Fig. 14—Worksheet for Born and Scidmore problem.

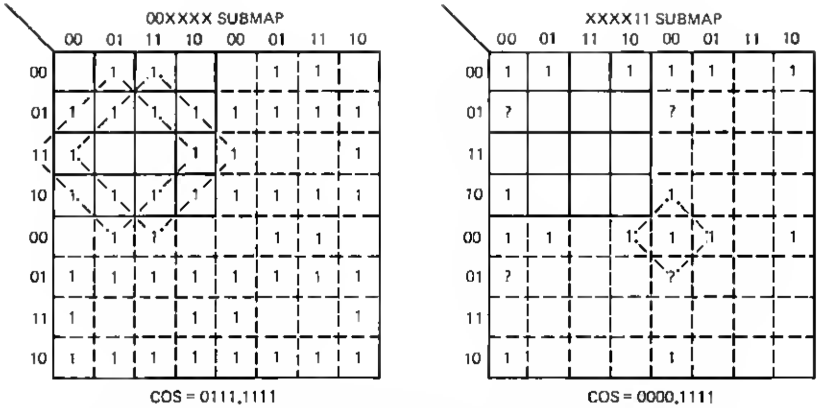

Fig. 15—Karnaugh submaps of Born and Scidmore problem.

submap 00XXXX (Fig. 16), and a single axis of symmetry (center 0111) is found.* The same last three digits are here as before, so the same terms appear on the XXXX11 submap except for the terms which came from $S_2^6(63)$.

If the three terms on the XXXX11 submap define a center of symmetry with a missing term, then this center of symmetry must be the same as before. Part of the minterms of $S_3^6(63)$, $S_4^6(63)$, and $S_1^6(63)$ appear, but none in sufficient quantity. Therefore, we assume that two of the terms on the XXXX11 submap are from extra terms and try each of the three points as a center of symmetry. Only one of these (1101) can be combined with 0111 so that a proposed center of symmetry is 110111. The terms are compared with it and 13 terms are found to match 110111 on two variables. Since 15 terms are required, the comparison is made for the missing terms. With experience, this can be done very readily. However, for completeness, the terms of $S_2^6(55)$ are tabulated in Fig. 17. The missing terms are 16 and 32. Thus, the function contains the function $S_2^6(55)$-16-32.

Looking at the 12 remaining terms we note that three of them (57, 58, 60) can be accounted for as $ABCS_1^3(DEF)$. The remaining nine terms all have the third variable in the zero state and so can be expressed as $C'f(ABDEF)$.

Repeating the technique on the nine-term, five-variable function results in the 0XXXX, 1XXXX, XXXX0, and XXXX1 submaps shown in Fig. 18. The first two submaps concur on a 1111 COS and the

* Note that the terms already identified appear as "don't care's" (D) in the submaps even though, in this case, they do not result in simplifying the remaining imbedded functions.

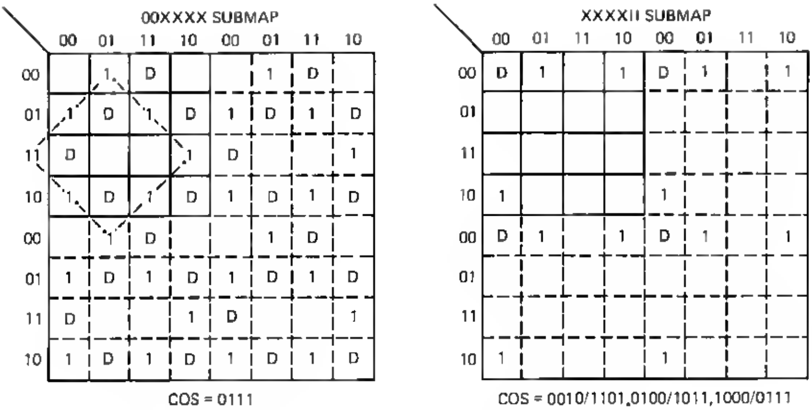

Fig. 16—Karnaugh submaps of remainder of function.

latter two submaps also result in a 1111 COS. As a result, a composite COS of 11111 is tried. Nine of the 10 needed terms for $S_3^5(31)$ are present with only the term 00111 missing. Thus, the original function has been broken up into the sum of four smaller functions, namely:

(*i*) $S_2^6(ABCDEF)$.
(*ii*) $S_2^6(ABC'DEF) - A'BC'D'E'F' - AB'C'D'E'F'$.
(*iii*) $ABCS_1^3(DEF)$.
(*iv*) $C'S_3^5(ABDEF) - A'B'C'DEF$.

TERM	N	INCLUDED IN FUNCTION
111000	56	✓
100000	32	
101100	44	✓
101010	42	✓
101001	41	✓
010000	16	
011100	28	✓
011010	26	✓
011001	25	✓
000100	4	✓
000010	2	✓
000001	1	✓
001110	14	✓
001101	13	✓
001011	11	✓

Fig. 17—Terms of symmetric S_2^6 (55).

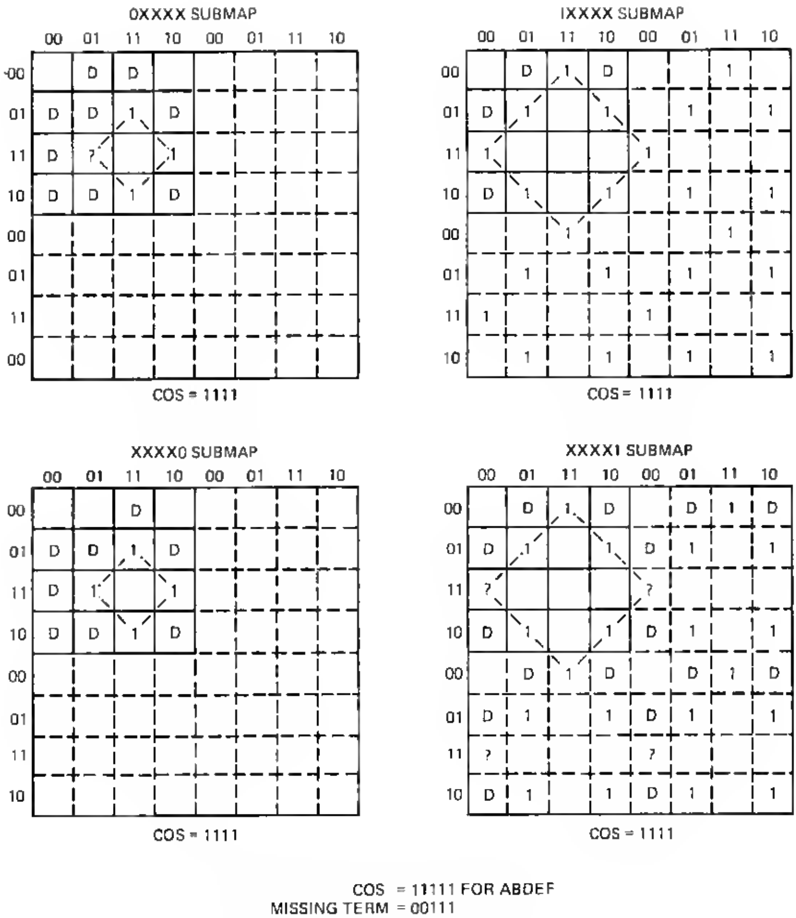

Fig. 18—Karnaugh submaps for final nine terms of function.

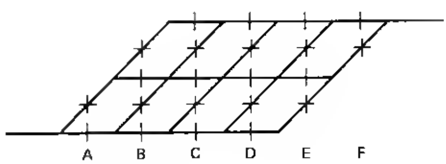

Fig. 19—Circuit for S_2^6 (63).

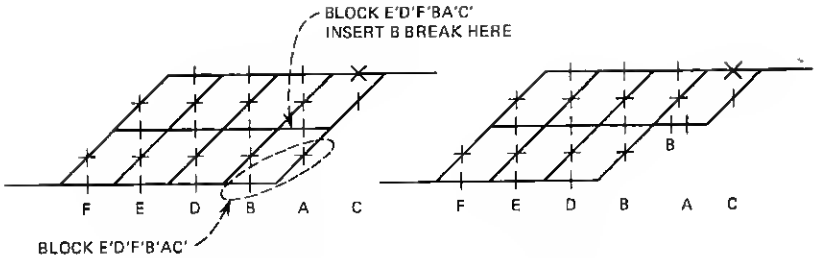

Fig. 20—Construction of circuit for $S_2^6(55)$-16-32.

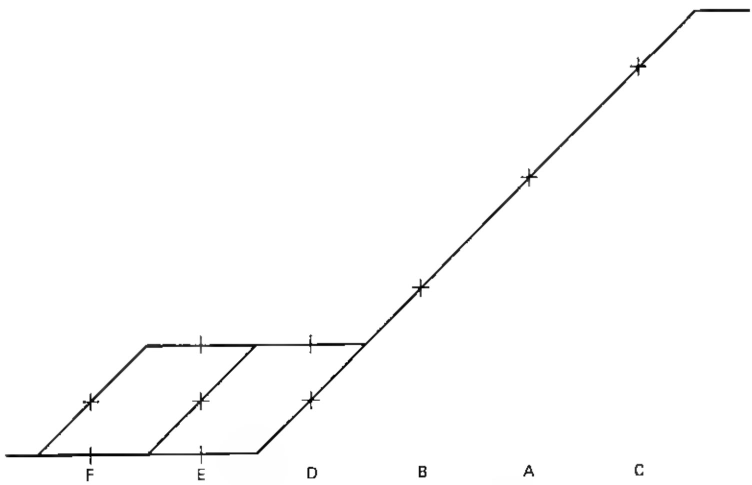

Fig. 21—Symmetric for $ABCS_1^3(DEF)$.

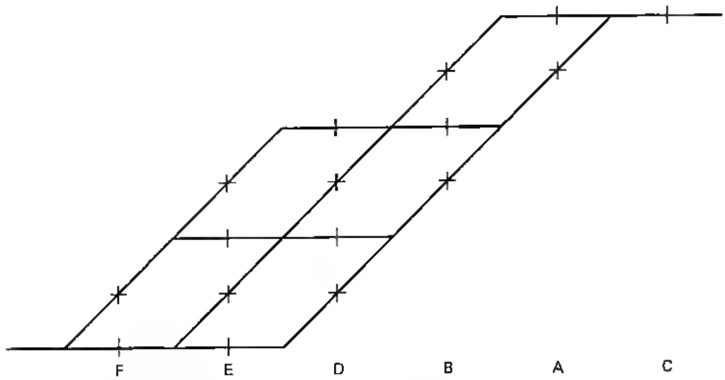

Fig. 22—Symmetric for $C'S_3^5(ABDEF)$—$A'B'C'DEF$.

RELAY	LOAD
A	3T
B	2T
C	1T
D	2T,1B
E	2T
F	1T
TOTAL	11T,1B

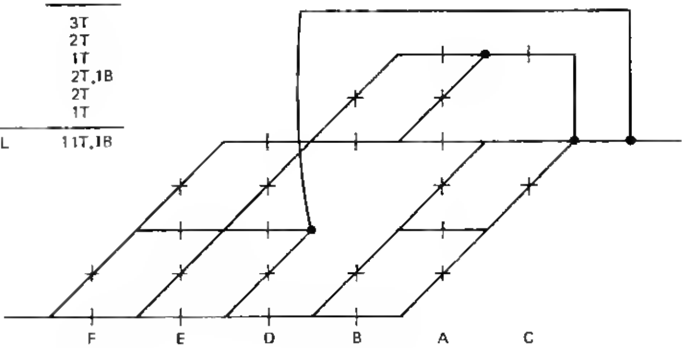

Fig. 23—Final circuit for Born and Scidmore problem.

These represent

(*i*) An SF.

(*ii*) An incomplete SF.

(*iii*) A minterm in three variables ANDed with an SF in the remaining three variables.

(*iv*) A single variable ANDed with an incomplete SF in the remaining five variables.

Figures 19 through 22 portray a relay configuration for each of these smaller functions, while Fig. 23 is the resulting circuit configuration when the preceding four circuits are combined. This resulting circuit uses 11 transfers and 1 break-contact, or less than half of the number required for the circuit resulting from the Born and Scidmore technique.

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